Vector Calculus
Line and Plane Summary

Dr. Paul L. Bailey
September 4, 2017

Lines in $\mathbb{R}^2$

A line in $\mathbb{R}^2$ is determined by a point $P_0 = (x_0, y_0)$ on the line, together with either a normal vector $\vec{n} = \langle a, b \rangle$, or a direction vector $\vec{v} = \langle v_1, v_2 \rangle$. Let $P = (x, y)$ denote an arbitrary point on the line.

Normal Vector

The vector from P_0 to P is perpendicular to the normal vector. This gives the first equation.

The *normal equation* of the line is

$$(P - P_0) \cdot \vec{n} = 0.$$

Now $P - P_0 = \langle x - x_0, y - y_0 \rangle$, so $\langle x - x_0, y - y_0 \rangle \cdot \langle a, b \rangle = 0$, whence $a(x - x_0) + b(y - y_0) = 0$. This gives the next equation.

The *general equation* of the line is

$$ax + by = c,$$

where $c = ax_0 + by_0$. If $b \neq 0$, we divide through by it to get the final form of the equation.

The *slope-intercept* form of the equation of the line is

$$y = mx + k,$$

where $m = -\dfrac{a}{b}$ and $k = \dfrac{c}{b}$.

Direction Vector

The vector $\vec{v}$ is in the direction of the line, so if we set $t = \dfrac{|P - P_0|}{|\vec{v}|}$, then $t\vec{v} = P - P_0$. Turn this around, and view P as a *function* of t. This gives the first equation.

The *vector equation* of the line is

$$P = P_0 + t\vec{v},$$

where P depends on t. Traditionally, if we view P as the path of a particle in motion, we set $\vec{r}(t) = P$, so that

$$\vec{r}(t) = P_0 + t\vec{v}.$$

We call $\vec{r}(t)$ the *position vector* of the particle, and we call $\vec{v}$ the *velocity vector*. Since $\vec{r}$ depends on t, so do the x and y coordinates of $\vec{r}$, so they are also functions of t, and we may write $\vec{r}(t) = \langle x(t), y(t) \rangle$. Thus $\langle x(t), y(t) \rangle = (x_0, y_0) + t\langle v_1, v_2 \rangle$. This leads to the next equations.

The *parametric equations* of the line are

$$x = x_0 + tv_1 \quad \text{and} \quad y = y_0 + tv_2.$$

Solving each of these equations for t and then equating the results give the next equation.

The *symmetric equation* of the line is

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2}.$$

Lines and Planes in $\mathbb{R}^3$

In three dimensions, only planes have normal vectors, and lines have direction vectors.

Normal Vector of a Plane

A plane in $\mathbb{R}^3$ is determined by a fixed point $P_0 = (x_0, y_0, z_0)$ on the plane, together with a normal vector $\vec{n} = \langle a, b, c\rangle$ for the plane.
Let $P = (x, y, z)$ denote an arbitrary point on the plane. The vector from P_0 to P is perpendicular to the normal vector. This gives the first equation.

The *normal equation* of the plane is

$$(P - P_0) \cdot \vec{n} = 0.$$

Now $P - P_0 = \langle x - x_0, y - y_0, z - z_0\rangle$, so

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0,$$

whence $a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$. This gives the next equation.

The *general equation* of the plane is

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Direction Vector of a Line

A line in $\mathbb{R}^3$ is determined by a point $P_0 = (x_0, y_0, z_0)$ on the line, together with a direction vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ for the line.
Let $P = (x, y, z)$ denote an arbitrary point on the line. The vector $\vec{v}$ is in the direction of the line, so if we set $t = \dfrac{|P - P_0|}{|v|}$, then $t\vec{v} = P - P_0$. Turn this around, and view P as a *function* of t. This gives the first equation.

The *vector equation* of the line is

$$P(t) = P_0 + t\vec{v},$$

or using the common "position vector" notation,

$$\vec{r}(t) = P_0 + t\vec{v}.$$

Since $\vec{r}$ depends on t, so do the x, y and z coordinates of $\vec{r}$, so they are also functions of t, and we may write $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. Thus $\langle x(t), y(t), z(t)\rangle = (x_0, y_0, z_0) + t\langle v_1, v_2, v_3\rangle$. This leads to the next equations.

The *parametric equations* of the line are

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad \text{and} \quad z = z_0 + tv_3.$$

Solving each of these equations for t and then equating the results give the next equations.

The *symmetric equations* of the line are

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v3}.$$